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THE ACTION OF A SOLVABLE GROUP ON AN INFINITE SET NEVER HAS A UNIQUE INVARIANT MEAN

STEFAN KRASA

ABSTRACT. Theorem 1 of the paper proves a conjecture of J. Rosenblatt on nonuniqueness of invariant means for the action of a solvable group G on an infinite set X. The same methods used in this proof yield even a more general result: Nonuniqueness still holds if G is an amenable group containing a solvable subgroup H such that $\operatorname{card}(G/H) \leq \operatorname{card}(H)$.

1. Introduction. Theorem 1 shows that for the action of a solvable group G on an infinite set X there can never exist a unique G-invariant mean on $l^\infty(X)$, where $l^\infty(X)$ denotes the space of all bounded real valued functions on X. A mean m on $l^\infty(X)$ is a positive linear functional on $l^\infty(X)$ with $m(1_X) = 1$, m is called a "G-invariant mean" if $m(gf) = m(f)$ for every $f \in l^\infty(X)$ and for every $g \in G$, where $gf(x) = f(g^{-1}(x))$. Theorem 2 is a generalization of Theorem 1 and proves nonuniqueness for the action of an amenable group G on X, where G contains a solvable subgroup H such that $\operatorname{card}(G/H) \leq \operatorname{card}(X)$. This paper is largely inspired by [**4**]. Proposition 6 of [**4**] shows that if G is nilpotent, then there will always exist more than one G-invariant mean on $l^\infty(X)$. Additionally it is stated in the same paper [**4**, p. 531]: "It seems to us very unlikely that Proposition 6 is false if G is solvable, but the same proof does not work". Nonuniqueness need not hold for the general amenable group action as Yang [**6**] recently showed. This solves a conjecture which is known as "Rosenblatt's problem" in the mathematical literature [**5**, p. 628]. The proof uses the continuum hypothesis. In Theorem 1 and Theorem 2 even more than nonuniqueness is proved: There exist two disjoint subsets Y_1, Y_2 of X such that $\bigcap_{i=1}^n g_i Y_k \neq \varnothing$ for $k = 1, 2$ and for every finite subset $\{g_1, \dots, g_n\}$ of G. If G does not act transitively on X, then there always exist many G-invariant means. The methods used in [**4**] show nonuniqueness if $\operatorname{card}(G) \leq \operatorname{card}(X)$. Obviously there exist examples of solvable groups G such that $\operatorname{card}(G) > \operatorname{card}(X)$ and such that G acts transitively on X. Example 3 of [**6**] even gives an example of a nilpotent group with this property.

EXAMPLE 1. We put $X = \bigoplus_{i=1}^\infty Z_i$ where $Z_i = \mathbf{Z}$ for every $i \in \mathbf{N}$. For $A \subset \mathbf{N}$ we define π_A as follows. If $x = (n_i)_{i \in \mathbf{N}}$ let $\pi_A(X) = (m_i)_{i \in \mathbf{N}}$ where $m_i = n_i$ if $i \notin A$ and $m_i = -n_i$ if $i \in A$. For $a \in X$ set $\tau_a(x) = a + x$ where addition is defined coordinatewise. Let G be the group generated by τ_a for all $a \in X$ and by π_A for every $A \subset \mathbf{N}$, then G is solvable and $\operatorname{card}(G) > \operatorname{card}(X)$.

In the proof, we use the following notations: $\operatorname{Per}(X)$ will denote the group of all invertible functions from X onto X. For $g \in \operatorname{Per}(X)$ and $Y \subset X$ we put $gY = \{g(y) : y \in Y\}$. For $C \subset \operatorname{Per}(X)$ we put $\mathcal{O}_C = \{C_x : x \in X\}$ to be the set of

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all C-orbits in X. For $F \subset \mathrm{Per}(X)$, and $Y \subset X$, we put $FY = \{fY : f \in F\}$. For $A, B \subset \mathrm{Per}(X)$, AB denotes the complex product. $\mathcal{C}$ will always denote the set of all cardinals c such that $c \leq \mathrm{card}(X)$.

2. Proof of the nonuniqueness of the mean. It is the essential problem of the following proof that we have to consider group actions where $\mathrm{card}(G) > \mathrm{card}(X)$. The rough idea is to split up the action of G into the action of a finite nested sequence of subsets P_i of $\mathrm{Per}(X)$ on P_i-orbits A_i such that $\mathrm{card}(P_i) \leq \mathrm{card}(A_i)$. Then we use the transfinite induction argument of Rosenblatt and Talagrand [**4**]. Unfortunately, if $\mathrm{card}(X)$ is an uncountable limit cardinal, some problems arise. In this case, we have to choose a family of P_i-orbits $A_m : m \in M$ with increasing cardinality where the index set M is an infinite subset of $\mathcal{C}$ with supremum $\mathrm{card}(X)$ such that $\mathrm{card}(A_m) = m$ for every $m \in M$. To each orbit A_m there will correspond a subset Q_m of $\mathrm{Per}(X)$ such that $\mathrm{card}(Q_m) \leq \mathrm{card}(A_m)$. For each A_m and Q_m we shall use the transfinite induction argument of [**4**]. Finally we make a transfinite induction on M to get the desired result. This does not work for an arbitrary M. However, if M is a discrete subset of $\mathcal{C}$, where $\mathcal{C}$ is endowed with the order topology, the argument can be done. It is proved in Lemma 2 that there always exists a discrete subset M of $\mathcal{C}$, which is still large enough for our purposes.

Before starting with the proof of Lemma 1, we state the following two well-known equalities for cardinals, we are going to use throughout the proof (cf. [**1**, p. 128]):

(i) $\aleph_\alpha^n = \aleph_\alpha$ for every ordinal α and for every $n \in N$.

(ii) $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha,\beta)}$ for all ordinals α, β. We also use standard results about amenable groups (cf. [**2**]). Throughout we assume that $\mathcal{C}$ is endowed with the order topology. We start with a technical lemma.

LEMMA 1. (i) *Let M be a subset of $\mathcal{C}$ with supremum s. Then there exists a discrete subset M' of M with supremum s.*

(ii) *Let M be a discrete subset of $\mathcal{C}$. For every $m \in M$ let $N(m)$ be a discrete subset of $\mathcal{C}$ with supremum m. Then for every $m \in M$ there exists a cardinal $c_m \in \mathcal{C}$ such that all the sets $N'(m) = \{n \in N(m) : n \geq c_m\}$ are nonempty and pairwise disjoint and such that $\bigcup_{m \in M} N'(m)$ is discrete.*

PROOF. (i) We can assume that $s \notin M$ otherwise we would choose $M' = \{s\}$. We now construct M' via transfinite induction on $M - \{s\}$. Assume that we have constructed for every $c < a$, $c, a \in M - \{s\}$ a discrete subset $M(c)$ of M such that $M(c)$ contains an element b with $b \geq c$ and such that for $c, c' \in M$, $c < c' < a$ implies that $M(c) \subset M(c')$. We put $N = \bigcup_{c \in M, c < a} M(c)$. If there exists an element m of N such that $m \geq a$ then we put $M(a) = N$ otherwise we choose a cardinal $b \in M$ such that $a < b < s$ and put $M(a) = N \cup \{b\}$. In both cases $M(a)$ is discrete. We define $M' = \bigcup_{m \in M, m < s} M(m)$. M' is a discrete subset of M and has s as supremum.

(ii) We use again a transfinite induction: Assume that we have constructed such a net of cardinals c_m for every $m \in M$ with $m < a$ where $a \in M$. Let $N = \bigcup_{m \in M, m < a} N(m)$, then $\mathrm{card}(N) < a$, because if $\mathrm{card}(N) = a$ then $a \in \overline{N}$, this together with $a \notin N$ implies that there does not exist a neighborhood V of a such that $V \cap M = \{a\}$ which is a contradiction to the discreteness of M. If $a \in N(a)$ we put $c_a = a$; otherwise we choose c_a such that $\mathrm{card}(N) < c_a < a$.

Definition 1 now defines a cardinality property we are going to need in Definition 2. It is the condition that allows us to use the transfinite induction argument (cf. [**4**]).

DEFINITION 1. Let $C \subset \mathrm{Per}(X)$ and $Y \subset X$. We say that a set $C' \subset \mathrm{Per}(X)$ is Y-complete with respect to C if C' is finite or $\mathrm{card}(C') \leq \mathrm{card}(Y)$ and if for every $c \in C$ there exists a $c' \in C'$ such that $c'(y) = c(y)$ for every $y \in Y$.

EXAMPLE 2. Let A be an abelian subgroup of $\mathrm{Per}(X)$ and let Y be an A-invariant subset of X such that A acts transitively on Y. Then there exists a subset A' of A which is Y-complete with respect to A.

We now give a definition, which we shall explain below.

DEFINITION 2. Let G be a subgroup of $\mathrm{Per}(X)$ and let $C_i\colon i \in \{1, \ldots, n\}$, D and E be subsets of $\mathrm{Per}(X)$. For every $i \in \{1, \ldots, n\}$ we put $P_i = C_i C_{i+1} \cdots C_n DE$. Furthermore we put $P_{n+1} = DE$ and $P_{n+2} = E$. We say that $(C_i\colon i \in \{1, \ldots, n\}, D, E)$ is a splitting of G if

(i) $G \subset P_1$,

(ii) for every $i \in \{1, \ldots, n+2\}$ and for every $A_1, A_2 \in \mathcal{O}_{P_i}$, $A_1 \cap A_2 \neq \emptyset$ implies that $A_1 = A_2$,

(iii) for every $i \in \{1, \ldots, n+2\}$ and for every $A \in \mathcal{O}_{P_i}$ we have $pA \subset A$ for every $p \in P_i$,

(iv) for every $i \in \{1, \ldots, n\}$ and for every $A \in \mathcal{O}_{P_i}$ there exists a set which is A-complete with respect to C_i,

(v) for every $A \in \mathcal{O}_E$ and for every $d \in D$, we have $dA \in \mathcal{O}_E$; and if $dA = d'A$ for every $A \in \mathcal{O}_E$, then $d = d'$.

Definition 2 introduces the groups for which we shall prove the nonuniqueness of the mean. Property (i) says that the group action is fully described by the family of subsets of $\mathrm{Per}(X)$, (ii) and (iii) guarantee that we have properly shaped orbits. (iv) is the cardinality property which is necessary for the transfinite induction argument. The sets D and E as well as property (v) will be used for the first time in the proof of Theorem 1, when we show that every solvable group admits a splitting where $D = E = \{e\}$. They are just defined in that way to keep the proof of Theorem 1 short.

LEMMA 2. *Let $(C_i\colon i \in \{1, \ldots, n\}, \{e\}, \{e\})$ be a splitting of G where G acts transitively on X. Then there exists a $j \in \{1, \ldots, n\}$ and a net of P_j-orbits $A_m\colon m \in M$ where $M \neq \emptyset$ such that*

(i) *M is a discrete subset of $\mathcal{C}$,*

(ii) *for every $m \in M$ we have $\mathrm{card}(A_m) = m$,*

(iii) *for every $i \in \{1, \ldots, j-1\}$ and for every $B \in \mathcal{O}_{P_i}$ such that $M_B \neq \emptyset$ where $M_B = \{m \in M\colon A_m \subset B\}$, we have $\mathrm{card}(\bigcup_{m \in M_B} A_m) = \mathrm{card}(B)$,*

(iv) *for every $m \in M$ there exists a cardinal $c \in \mathcal{C}$ such that for every P_{j+1}-orbit B which is contained in A_m, $\mathrm{card}(B) \leq c < m$ holds.*

PROOF. We reformulate Lemma 2 such that an induction argument can be applied.

Claim 1. Let $(C_i\colon i \in \{1, \ldots, n\}, \{e\}, \{e\})$ and G be as above and let $t \in \{1, \ldots, n\}$; then there exists a $j \leq t$ and a net of P_j-orbits $A_m\colon m \in M$ such that either $j \leq t$ and (i)–(iv) of Lemma 2 hold or $j = t$ and (i)–(iii) of Lemma 2 hold.

We prove Claim 1 by induction on t: For $t = 1$ there is nothing to prove. We assume therefore that Claim 1 holds for $t < n$ and give the proof for $t+1$.

Let $A_m : m \in M$ be the net of P_j-orbits. We can assume that $j = t$, otherwise we would have the proof for $t+1$. The essential idea of the following is the construction of a net of P_{t+1}-orbits $B_n : n \in N(m)$ in each A_m such that $\text{card}(B_n) = n$ for every $n \in N(m)$ and such that $\text{card}(\bigcup_{n\in N(m)} B_n) = m$. Lemma 1(i) and (ii) then complete the proof. The sets $B_n : n \in N(m)$ however can only be constructed if Claim 2 from below holds. Claim 3 proves that we can assume without loss of generality that Claim 2 holds.

Claim 2. For every $m \in M$ and for every cardinal c such that $c < m$ there exists a P_{t+1}-orbit B such that $B \subset A_m$ and $c < \text{card}(B) \le m$.

Claim 3. Let $M_1 = \{m \in M$: for every cardinal $c < m$ there exists a P_{t+1}-orbit $B \subset A_m$ such that $c < \text{card}(B) \le m\}$ and let $M_2 = M - M_1$. Then for every $k \in \{1, \ldots, t\}$ and for every P_k-orbit B such that $M_B \ne \emptyset$ there exists a net $A_n : n \in N$ of P_t-orbits which are contained in B such that

(i) either $N \subset M^1$ or $N \subset M^2$,

(ii) for every $i \in \{k, \ldots, t\}$ and for every P_i-orbit B' such that $N_{B'} \ne \emptyset$ we have $\text{card}(\bigcup_{n\in N_{B'}} A_n) = \text{card}(B')$.

We prove Claim 3 by induction on k. For $k = t$ there is nothing to prove. We now assume that Claim 3 holds for $k > 1$ and give the proof for $k-1$.

Let B be a P_{k-1}-orbit such that $M_B \ne \emptyset$. Then there exists a net $B_b : b \in \mathcal{B}$ of P_k-orbits such that $\text{card}(B_b) = b$, such that $M_{B_b} \ne \emptyset$ for every $b \in \mathcal{B}$ and such that $\mathcal{B}$ has $\text{card}(B)$ as supremum. We fix $b \in \mathcal{B}$; then there exists a net $A_n : n \in N(b)$ such that (ii) holds and such that either $N(b) \subset M^1$ or $N(b) \subset M^2$. Let $\mathcal{B}^1$ be the subset of $\mathcal{B}$ where $N(b) \subset M^1$ and $\mathcal{B}^2$ the subset of $\mathcal{B}$ where $N(b) \subset M^2$. Then because $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$ either $\mathcal{B}^1$ or $\mathcal{B}^2$ has $\text{card}(B)$ as supremum in $\mathcal{C}$. Lemma 1(i) and (ii) applied to that $\mathcal{B}^i$ yields the existence of a net $A_n : n \in N$ such that (i) and (ii) of Claim 3 hold for B. This, however, implies Claim 3.

For $k = 1$ Claim 3 implies that we can assume for the proof of Claim 1 that Claim 2 holds: The net $A_n : n \in N$ fulfils (i)–(iii) of Lemma 2 and also (iv) of Lemma 2 if $N \subset M^2$ which would conclude the proof of Claim 1. Therefore we can assume that $N \subset M^1$ which means that Claim 2 holds.

We now fix a $m \in M$. Claim 2 and Lemma 1(i) imply the existence of a net of P_{t+1}-orbits $B_n : n \in N(m)$ such that $B_n \subset A_m$ and $\text{card}(B_n) = n$ for every $n \in N(m)$ and such that $N(m)$ is a discrete subset of $\mathcal{C}$ with supremum m. Lemma 1(ii) now implies the existence of a net $B_n : n \in N$ of P_{t+1}-orbits such that (i)–(iii) of Lemma 2 hold. This proves Claim 1. For $t = n$ Claim 1 implies Lemma 2.

We are now able to show the nonuniqueness of the mean for amenable groups which have a splitting where $D = E = \{e\}$ by proving the existence of two disjoint subsets A_1, A_2 such that $\bigcap_{i=1}^n g_i A_k \ne \emptyset$ for $k = 1, 2$ and for every finite subset $\{g_1, \ldots, g_n\}$ of G. Claim 4 contains the transfinite induction argument of **[4]**. It is the proof of the induction step of Claim 5. The proof of Claim 5 is the transfinite induction on M which together with Claim 6 yields the complete result.

LEMMA 3. *Let G be an amenable subgroup of* $\text{Per}(X)$ *and let $(C_i : i \in \{1, \ldots, n\}$, $\{e\}, \{e\})$ be a splitting of G. If X is an infinite set then there exist many G-invariant means on $l^\infty(X)$.*

PROOF. We can assume that G acts transitively on X otherwise if Y_1 and Y_2 are two different G-orbits then there exist two G-invariant means m_1 and m_2 on $l^\infty(X)$ such that $m(1_{Y_k}) = 1$ for $k = 1, 2$ (cf. [2, pp. 16, 17] or [3, Lemma 1]) which implies that $m_1 \neq m_2$.

We can therefore take the net $A_m := m \in M$ of Lemma 2. We are now going to construct to each A_m a suitable subset Q_m of $\mathrm{Per}(X)$ such that $\mathrm{card}(Q_m) \leq \mathrm{card}(A_m)$ which implies that for each $m \in M$ the transfinite induction argument can be applied and such that a subsequent induction on M yields the complete result.

We now consider the case where $\mathrm{card}(X) > \aleph_0$. Let $A_m : m \in M$ be the net of P_j-orbits. We can assume that $m \geq \aleph_0$ for every $m \in M$ because the net $A_m : m \in M'$ where $M' = \{m \in M : m \geq \aleph_0\}$ also fulfils (i)–(iv) of Lemma 2.

Let B^k be the collection of all P_k-orbits which contain at least one A_m; then $B^1 = \{X\}$ because G acts transitively. Let H_X be X-complete with respect to C_1. For every $B \in B^2$ we now choose a subset H_X^B of H_X such that $\mathrm{card}(H_X^B) \leq \mathrm{card}(B)$, such that $H_X^{B_1} \subset H_X^{B_2}$ if $\mathrm{card}(B_1) \leq \mathrm{card}(B_2)$, and such that $\bigcup_{B \in B^2} H_X^B = H_X$. We can do that because of (iii) of Lemma 2. Let H_B be a B-complete subset of C_2. We put $H_{X,B} = H_X^B H_B$; then $\mathrm{card}(H_{X,B}) \leq \mathrm{card}(B)$. For $B \in B^2$ let $B_B^3 = \{B' \in B^3 : B' \subset B\}$. Now we choose for every $B' \in B_B^3$ a subset $H_{X,B}^{B'}$ such that $\mathrm{card}(H_{X,B}^{B'}) \leq \mathrm{card}(B')$, such that $H_{X,B}^{B_1'} \subset H_{X,B}^{B_2'}$ if $\mathrm{card}(B_1') \leq \mathrm{card}(B_2')$, and such that $\bigcup_{B' \in B_B^3} H_{X,B}^{B'} = H_{X,B}$ and put $H_{X,B,B'} = H_{X,B}^{B'} H_{B'}$ where $H_{B'}$ is B'-complete with respect to C^3. We continue this construction until we reach the P_j-orbits A_m. We put $Q_m = H_{X,B^2,B^3,\ldots,B^{j-1},A_m}$. Then $\mathrm{card}(Q_m) \leq m$.

If $\mathrm{card}(X) = \aleph_0$ the construction is much easier. Instead of considering the net $A_m : m \in M$ of Lemma 2, we choose a P_j-orbit B where B is countable and where j is the largest index such that a countable orbit exists. We then put $A_{\aleph_0} = B$ and choose $A_m : m \in \{\aleph_0\}$ as net of P_j-orbits. Finally we choose for $Q_{\aleph_0}$ a subset of $\mathrm{Per}(X)$ which is $A_{\aleph_0}$-complete with respect to $C_1 C_2 \cdots C_j$.

Claim 4. Let $m \in M$ and $W \subset X$ such that $\mathrm{card}(W) < m$. Let $F_m = \{F \subset Q_m : F \text{ finite}\}$ and let $F_\alpha : \alpha < \delta_m$ be an enumeration of F_m where δ_m is the least ordinal such that $\mathrm{card}(\delta_m) = \mathrm{card}(Q_m) \leq m$. Then there exists a net of P_{j+1}-orbits $P_\alpha^k : \alpha < \delta_m$, $k = 1, 2$, which are contained in A_m such that all sets $F_\alpha P_\alpha^k : \alpha < \delta_m$, $k = 1, 2$, together with W are pairwise disjoint.

We prove this with transfinite induction. Let $\beta < \delta_m$ and assume that we have constructed such a net for every $\alpha < \beta$. We now construct P_{j+1}-orbits P_β^1 and P_β^2. Let $V = W \cup \bigcup_{\alpha < \beta} F_\alpha P_\alpha^1 \cup \bigcup_{\alpha < \beta} F_\alpha P_\alpha^2$. Then if $\mathrm{card}(X) = \aleph_0$, we get $\mathrm{card}(\bigcup_{\alpha < \beta} F_\alpha P_\alpha^k) < \aleph_0$ for $k = 1, 2$; and if $\mathrm{card}(X) > \aleph_0$, there exists (because of (iv) of Lemma 2) a cardinal c such that $\mathrm{card}(P_\alpha^k) \leq c < m$ for every $\alpha < \beta$ and for $k = 1, 2$, consequently we get $\mathrm{card}(\bigcup_{\alpha < \beta} F_\alpha P_\alpha^k) \leq \max(c, \mathrm{card}(\beta)) < m$ for $k = 1, 2$. Therefore we get in both cases $\mathrm{card}(V) < m$. Let $F_\beta = \{q_1, \ldots, q_r\}$, then $\mathrm{card}(\bigcup_{i=1}^r q_i^{-1} V) < m$. If $\mathrm{card}(X) = \aleph_0$, A obviously has to contain $\aleph_0$-many P_{j+1}-orbits, if $\mathrm{card}(X) > \aleph_0$ property (iv) of Lemma 2 implies the existence of m-many P_{j+1}-orbits in A_m. In both cases there therefore has to exist a P_{j+1}-orbit P_β^1, such that $P_\beta^1 \subset A_m - \bigcup_{i=1}^r q_i^{-1} V$ which implies that $P_\beta^1 \subset A_m$ and that $F_\beta P_\beta^1$, W together with all sets $F_\alpha P_\alpha^k : \alpha < \beta$, $k = 1, 2$, are pairwise disjoint. We now put

$V' = V \cup F_\beta P_\beta^1$ and repeat the construction from above to get a P_{j+1}-orbit P_β^2. This proves Claim 4.

Claim 5. Let $A_m: m \in M$ and $Q_m: m \in M$ be as above; then there exist subsets Y_1, Y_2 of X, $Y_1 \cap Y_2 = \emptyset$ such that for every $m \in M$ and for every finite subset $\{q_1, \dots, q_r\}$ of Q_m there exist P_{j+1}-orbits Z_1, Z_2 such that $Z_1, Z_2 \subset A_m$ and $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$.

We prove Claim 5 by induction on M. Let m_0 be the least cardinal of M. Let $W = \emptyset$ and $m = m_0$. Then Claim 4 implies Claim 5 for $m = m_0$ if we put $Y_k(m_0) = \bigcup_{\alpha < \delta_{m_0}} F_\alpha P_\alpha^k$ for $k = 1, 2$. We now assume that Claim 5 holds for every $n < m$; $n, m \in M$. That means for every $n < m$ there exists a subset $Y_k(n)$ of X, $\operatorname{card}(Y_k(n)) \leq n$ such that Claim 5 holds for every finite subset of $Q_{n'}$, where $n' \leq n$ and that $Y_k(n) \subset Y_k(n')$ for $n \leq n' < m$ and for $k = 1, 2$. Let $W = \bigcup_{n<m} Y_1(n) \cup \bigcup_{n<m} Y_2(n)$; then $\operatorname{card}(W) < m$ because $\operatorname{card}(Y_k(n)) \leq n$ for $n \in M$, $n < m$ and for $k = 1, 2$ and because there exists a neighborhood V of m such that $V \cap M = \{m\}$ due to the discreteness of M. We now use Claim 4 to conclude the proof of Claim 5.

Claim 6. If there exist subsets Y_1, Y_2 of X such that for every $m \in M$ and for every finite subset $\{q_1, \dots, q_r\}$ of Q_m there exist P_{j+1}-orbits $Z_1, Z_2 \subset A_m$ such that $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$ then for every finite subset $\{h_1, \dots, h_r\}$ of P_1 there exist $x_1, x_2 \in X$ such that $\bigcup_{i=1}^r h_i(x_k) \subset Y_k$ for $k = 1, 2$.

Let $h_i = c_{i,1} c_{i,2} \cdots c_{i,j} b_i$ where $c_{i,k} \in C_k$ and $b_i \in P_{j+1}$. Then there exists a $c'_{i,1} \in H_X$ such that $c'_{i,1}(x) = c_{i,1}(x)$ for every $x \in X$. There exists a $B \in B^2$ such that $c'_{i,1} \in H_X^B$ for every $i \in \{1, \dots, r\}$. Let $c'_{i,2} \in H_B$ such that $c'_{i,2}(x) = c_{i,2}(x)$ for every $x \in B$. Then $c'_{i,1} c'_{i,2} \in H_{X,B}$. Therefore there exists a P_3-orbit $B' \subset B$ such that $c'_{i,1} c'_{i,2} \in H_{X,B}^{B'}$ for every $i \in \{1, \dots, r\}$. We continue this construction until we reach a P_j-orbit A_m. Then there exists a subset $\{q_1, \dots, q_r\}$ of Q_m such that $h_i(x) = q_i b_i(x)$ for every $x \in A_m$ and for every $i \in \{1, \dots, r\}$. Let Z_1, Z_2 be P_{j+1}-orbits such that $Z_1, Z_2 \subset A_m$ and $\bigcup_{i=1}^r q_i Z_k \subset Y_k$ for $k = 1, 2$, then $q_i Z_k = c_{i,1} c_{i,2} \cdots c_{i,j} Z_k$ for $k = 1, 2$ and $i \in \{1, \dots, r\}$; and because of (iii) of Definition 2, we get $b_i Z_k \subset Z_k$. This together implies Claim 6.

Let Y_1, Y_2 be the disjoint subsets of Claim 5 and let $\{g_1, \dots, g_r\}$ be an arbitrary finite subset of G. Then $\{g_1^{-1}, \dots, g_r^{-1}\} \subset P_1$ and because of Claim 5 and Claim 6 there exist $x_1, x_2 \in X$ such that $\bigcup_{i=1}^r g_i^{-1}(x_k) \subset Y_k$ for $k = 1, 2$. This however implies that $\bigcap_{i=1}^r g_i Y_k \neq \emptyset$ for $k = 1, 2$, which implies the existence of two G-invariant means m_1 and m_2 on $l^\infty(X)$ such that $m_k(1_{Y_k}) = 1$ for $k = 1, 2$ (cf. [**3**, Lemma 2]). As Y_1 and Y_2 are disjoint m_1 and m_2 have to be two different G-invariant means.

THEOREM 1. *Let G be a solvable group which acts on the infinite set X; then there exist many G-invariant means on $l^\infty(X)$.*

PROOF. It is well known that every solvable group is amenable (cf. [**2**]). Because of Lemma 1 it now remains to prove that there exists a splitting $(C_i: i \in \{1, \dots, n\}, \{e\}, \{e\})$ of G.

Claim 7. Let $(C_i: i \in \{1, \dots, n\}, D, E)$ be a splitting of G. Because of (ii) and (v) of Definition 2 we can regard D to be a subset of $\operatorname{Per}(\mathcal{O}_E)$. We assume that D is a solvable subgroup of $\operatorname{Per}(\mathcal{O}_E)$. Then there exists a splitting $(C_i: i \in \{1, \dots, n'\}, \{e\}, E)$ of G.

We prove Claim 7 by induction on the length of the commutator chain of D. For $D = \{e\}$ there is nothing to prove. Assume that Claim 7 holds for every D with a commutator chain of length $k \leq n$ and assume that D has a commutator chain of length $n+1$ that means $D = D_0 \trianglerighteq D_1 \trianglerighteq \cdots \trianglerighteq D_n \trianglerighteq D_{n+1} = \{e\}$. Then $(C_i\colon i \in \{i,\dots,n\},\ D/D_n,\ [\bigcap_{A\in\mathcal{O}_E} T_D(D_nA)]E)$ is a splitting of G where D/D_n is considered to be a subgroup of $\operatorname{Per}(\mathcal{O}_{D_nE})$ and where $T_D(D_nA) = \{d \in D\colon d(D_nA) = D_nA\}$. As Claim 7 holds for n we therefore get a splitting $(C_i\colon i \in \{1,\dots,n'\},\ \{e\},\ [\bigcap_{A\in\mathcal{O}_E} T_D(D_nA)]E)$ of G which implies that $(C_i\colon i \in \{1,\dots,n'\},\ \bigcap_{A\in\mathcal{O}_E} T_D(D_nA), E)$ is a splitting of G. We define $H_1 = \{h \in \operatorname{Per}(X)\colon$ for every $A \in \mathcal{O}_{D_nE}$ there exists $d_A \in D_n$ such that $d_A(x) = h(x)$, for every $x \in A\}$. Let $A_i\colon i \in I$ be a collection of E-orbits such that each D_nE-orbit contains exactly one of the sets A_i. For every $i \in I$ let S_i be a system of representatives for the cosets of $T_D(A_i)/(T_D(A_i) \cap D_n)$ where $T_D(A_i) = \{d \in D\colon d(A_i) = A_i\}$. S_i acts as a solvable group with a commutator chain of length $k \leq n$ on the E-orbits contained in D_nA_i. We now define $H_2 = \{h \in \operatorname{Per}(X)\colon$ for every $i \in I$ there exists a $s_i \in S_i$ such that $s_i(x) = h(x)$ for every $x \in D_nA_i\}$, then $\bigcap_{A\in\mathcal{O}_E} T_D(D_nA) \subset H_1H_2$. H_2 can be embedded into the group $\prod_{i\in I} S_i$ which is solvable with a commutator chain of length $k \leq n$. H_1 is an abelian group which acts transitively on every set D_nA_i; therefore Example 2 implies that for every $A \in \mathcal{O}_E$ there exists a set which is D_nA-complete with respect to H_1. We put $C_{n'+1} = H_1$ then $(C_i\colon i \in \{1,\dots,n'+1\}, H_2, E)$ is a splitting of G, as H_2 has a commutator chain of length $k \leq n$ there has to exist a splitting $(C_i\colon i \in \{1,\dots,n''\}, \{e\}, E)$ of G which implies Claim 7.

Obviously $(\{e\}, G, \{e\})$ is a splitting of G. Claim 7 implies the existence of a splitting $(C_i\colon i \in \{1,\dots,n\}, \{e\}, \{e\})$ of G which yields because of Lemma 3 the nonuniqueness of the mean.

REMARK. We could simplify the proof of the result from above a lot, if we assume that $\operatorname{card}(X)$ is no limit cardinal, because then all the nets $A_m\colon m \in M$ could be reduced to nets which contain only one element. Such an assumption, however, would not be justified as it is easy to construct examples where $\operatorname{card}(X)$ is a limit cardinal, G is solvable and $\operatorname{card}(G) > \operatorname{card}(X)$ by similar methods as in Example 1.

We can even prove a slight generalization of Theorem 1.

THEOREM 2. *Let G be an amenable group which acts on the infinite set X. Let H be a solvable subgroup of G such that* $\operatorname{card}(G/H) \leq \operatorname{card}(X)$. *Then there exist many G-invariant means on $l^\infty(X)$.*

PROOF. Let C be a system of representatives for the cosets of G/H. Then $(C, H, \{e\})$ is a splitting of G. Claim 7 and Lemma 3 imply the nonuniqueness.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIENNA, A-1090 VIENNA, AUSTRIA